

# Strong solutions and weak-strong stability in a system of cross-diffusion equations

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## Abstract

Proving the uniqueness of solutions to multi-species cross-diffusion systems is a difficult task in the general case, and there exist very few results in this direction. In this work, we study a particular system with zero-flux boundary conditions for which the existence of a weak solution has been proven in [14]. Under additional assumptions on the value of the cross-diffusion coefficients, we are able to show the existence of strong solutions. The proof relies on the use of an appropriate approximation and a fixed-point argument. In addition, a weak-strong stability result is obtained for this system in dimension one which implies uniqueness.

## 1 Introduction

Systems of partial differential equations with cross-diffusion have gained a lot of interest in recent years [23, 8, 9, 24, 22]. Such systems appear in many applications, for instance the modelling of population dynamics of multiple species [5], cell sorting or chemotaxis-like applications [28, 27] or predator-swarm model [10], and have been studied in different contexts [1, 25, 17, 16, 6, 7].

In this work we focus our attention to a particular multi-species cross-diffusion system which reads as follows. Let  $T > 0$ ,  $n, d \in \mathbb{N}^*$  and  $\Omega \subset \mathbb{R}^d$  be a bounded regular domain. For  $t \in (0, T)$  and  $x \in \Omega$ , we consider  $(u_0(t, x), \dots, u_n(t, x))$  to be a solution to the system of  $n + 1$  equations

$$\partial_t u_i - \nabla \cdot \left[ \sum_{j=0, j \neq i}^n K_{ij} (u_j \nabla u_i - u_i \nabla u_j) \right] = 0 \text{ in } (0, T) \times \Omega, \quad i = 0, \dots, n, \quad (1)$$

supplemented with no-flux boundary conditions and some initial data, where  $K_{ij} \geq 0$  for all  $0 \leq i \neq j \leq n$ .

System (1) can be seen as the (formal) limit of a microscopic stochastic lattice based model (see the Appendix for more details) and models the evolution of the local volumic fractions of a system composed of  $n + 1$  different species. The function  $u_i(t, x)$  represents the value at some time  $t \in [0, T]$  and point  $x \in \Omega$  of the density or volumic fraction of the  $i^{\text{th}}$  entity. In terms of modelling this means

that the particles whose densities are given by the functions  $u_i$  have a finite, positive size so that there is a maximal number of particles per given volume. This is often referred to as size exclusion (or exclusion process). From a modelling point of view, one is therefore interested in considering solutions to (1) which satisfy

$$\forall 0 \leq i \leq n, \quad u_i(t, x) \geq 0 \quad \text{and} \quad \sum_{i=0}^n u_i(t, x) = 1, \quad \text{a.e. in } (0, T) \times \Omega. \quad (2)$$

From an analysis point of view, it is not easy to prove the existence of solutions to cross-diffusion systems satisfying (2), and uniqueness results are even harder to obtain. Recently, cross-diffusion systems which exhibit a (formal) gradient flow structure (see [20], [4], or [31]) have drawn particular interest from mathematicians. Indeed, such a structure allows to show the existence of weak solutions in many situations, using the dissipation of the corresponding entropy to get a priori bounds which are enough to pass to the limit in a suitable approximation. This often also relies on the introduction of so-called entropy variables which can be used as a substitute for maximum principles which are not available for such systems. See e.g. [19], [20] and [14] for examples of this strategy. Also note that due to the degenerate structure of (1), solutions sometimes have less regularity than in the usual parabolic case (e.g. for  $n = 2$ ,  $K_{10} = K_{20} = 1$  and  $K_{12} = K_{21} = 0$ , the solutions  $u_i$  are only  $L^2$  in space, not  $H^1$ , see [4] for details).

The existence of weak solutions to (1) is proved in [14], using a general result of [22], under the assumption that the cross-diffusion coefficients  $(K_{ij})_{0 \leq i \neq j \leq n}$  are assumed to be positive and to satisfy  $K_{ij} = K_{ji}$  for all  $0 \leq i \neq j \leq n$ . Most importantly in the context of our work, the existence of strong solutions and uniqueness of weak solutions was so far only available in a very special situation, i.e. when all the self-diffusion coefficients  $K_{ij}$  are equal to a constant  $K$ . In this case, system (1) boils down to a system of  $n + 1$  independent heat equations, the analysis of which is easy. In the general case, to the best of our knowledge, no such results are available so far.

The object of the present article is based on the following observation: The system (1) can be considered as a perturbation of a system of heat equations if the coefficients  $K_{ij}$  are not too different from a fixed constant  $K$ . Indeed, we have

$$\partial_t u_i - K \Delta u_i = \operatorname{div} \left[ \sum_{j=0}^n (K_{ij} - K) (u_j \nabla u_i - u_i \nabla u_j) \right]. \quad (3)$$

Under the assumption that the quantities  $|K_{ij} - K|$  are sufficiently small, we prove the existence of strong solutions to (1), along with a weak-strong stability estimate in dimension 1 which implies the uniqueness of the solution. A key issue in the proof is to construct approximations that preserve nonnegativity and the volume constraint.

This paper is organized as follows: in Section 2 we state our main results. Section 3 is devoted to the proof of the existence of strong solutions to the cross-diffusion system we consider. Lastly, Section 4 details the proof of the weak-strong stability result we obtain in dimension 1. Let us mention that a weak-strong stability result is proved in [11] for a system similar to (1), but with different assumptions on the coefficients  $(K_{ij})_{0 \leq i \neq j \leq n}$ .

## 2 Main results

### 2.1 Notation and preliminaries

Let  $T > 0$ ,  $n, d \in \mathbb{N}^*$  and  $\Omega \subset \mathbb{R}^d$  be a bounded regular domain. For  $t \in (0, T)$  and  $x \in \Omega$ , we consider  $u(t, x) := (u_0(t, x), \dots, u_n(t, x))$  solution to the system of  $n + 1$  equations:

$$\begin{cases} \partial_t u_i - \nabla \cdot \left[ \sum_{j=0, j \neq i}^n K_{ij} (u_j \nabla u_i - u_i \nabla u_j) \right] = 0 & \text{in } (0, T) \times \Omega, \\ \left[ \sum_{j=0, j \neq i}^n K_{ij} (u_j \nabla u_i - u_i \nabla u_j) \right] \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad i = 0, \dots, n, \quad (4)$$

where  $\mathbf{n}$  denotes the unit outward pointing normal to the domain  $\Omega$ , and  $(K_{ij})_{0 \leq i \neq j \leq n}$  are non-negative coefficients. System (4) is supplemented with the initial condition  $u^0 := (u_0^0, \dots, u_n^0) \in (L^1(\Omega))^{n+1}$

We make the following assumption on the values of the cross-diffusion coefficients  $(K_{ij})_{0 \leq i \neq j \leq n}$ :

**Assumption 1.** For all  $0 \leq i \neq j \leq n$ ,  $K_{ij} > 0$  and  $K_{ij} = K_{ji}$ .

As mentioned in the introduction, such a system models the evolution of the local volumic fractions of a system composed of  $n + 1$  different species and we expect the nonnegativity and volume constraint (2) to hold.

Let us denote by

$$\mathcal{P} := \left\{ u := (u_0, \dots, u_n) \in (0, +\infty)^{n+1}, \sum_{i=0}^n u_i = 1 \right\} \quad \text{and} \quad \mathcal{D} := \left\{ U := (u_1, \dots, u_n) \in (0, +\infty)^n, \sum_{i=1}^n u_i < 1 \right\}.$$

We point out that for all  $u := (u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ ,  $u$  belongs to  $\mathcal{P}$  if and only if  $U := (u_1, \dots, u_n)$  belongs to  $\mathcal{D}$ . Similarly,  $u$  belongs to  $\overline{\mathcal{P}}$  if and only if  $U$  belongs to  $\overline{\mathcal{D}}$ . Let us mention that condition (2) can be equivalently rewritten as  $u(t, x) \in \overline{\mathcal{P}}$  for almost all  $(t, x) \in (0, T) \times \Omega$ . In what follows, we assume that the initial condition  $u^0$  satisfies the following constraint:

$$u^0(x) \in \overline{\mathcal{P}} \quad \text{for almost all } x \in \Omega. \quad (5)$$

Under Assumptions 1 and 5, it is easy to see (at least formally) that the dynamics of the system preserves the volume constraint, i.e.

$$\sum_{i=0}^n u_i(t, x) = 1 \quad \text{a.e. in } (0, T) \times \Omega. \quad (6)$$

However, proving the existence of (weak or strong) solutions to system (4) so that  $u_i(t, x) \geq 0$  for all  $0 \leq i \leq n$  and almost all  $(t, x) \in (0, T) \times \Omega$  is an intricate task from an analysis point of view. The existence of weak solutions to system (4) satisfying (2) is proved in [14] under Assumptions 1 and 5 and is actually a consequence of Theorem 2 of [22]. Let us recall this result and the main arguments of its proof below. Using (6) to express  $u_0$  as  $1 - \sum_{i=1}^n u_i$ , system (4) can be equivalently rewritten as a system of  $n$  equations of the form

$$\begin{cases} \partial_t U - \nabla \cdot (A(U) \nabla U) = 0 & \text{on } (0, T) \times \Omega, \\ (A(U) \nabla U) \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (7)$$

where  $U := (u_1, \dots, u_n)$ . The diffusion matrix  $A$  is defined by

$$A : \begin{cases} \mathbb{R}^n & \rightarrow & \mathbb{R}^{n \times n} \\ U & \mapsto & A(U) := (A_{ij}(U))_{1 \leq i, j \leq n}, \end{cases} \quad (8)$$

with, for all  $U := (u_1, \dots, u_n) \in \mathbb{R}^n$ ,

$$\begin{aligned} A_{ii}(U) &= \sum_{j=1, j \neq i}^n (K_{ij} - K_{i0})u_j + K_{i0}, & i = 1, \dots, n, \\ A_{ij}(U) &= -(K_{ij} - K_{i0})u_i, & i, j = 1, \dots, n, i \neq j. \end{aligned} \quad (9)$$

Theorem 2 of [22] gives sufficient conditions on the diffusion matrix  $A$  for a general cross-diffusion system to have a weak solution so that  $U(t, x) \in \overline{\mathcal{D}}$  for almost all  $(t, x) \in (0, T) \times \Omega$ . More precisely, Theorem 2 of [22] can be stated as follows.

**Theorem 2.1** (Theorem 2 of [22]). *Let  $A \in \mathcal{C}^0(\overline{\mathcal{D}}; \mathbb{R}^{n \times n})$  be a continuous matrix-valued field defined on  $\overline{\mathcal{D}}$  satisfying the following assumptions:*

(H1) *there exists a bounded from below convex function  $h \in \mathcal{C}^2(\mathcal{D}, \mathbb{R})$  such that its derivative  $Dh : \mathcal{D} \rightarrow \mathbb{R}^n$  is invertible on  $\mathbb{R}^n$ ;*

(H2) *there exists  $\alpha > 0$ , and for all  $1 \leq i \leq n$  there exists  $1 \geq m_i > 0$  such that, for all  $z := (z_1, \dots, z_n) \in \mathbb{R}^n$  and all  $U := (u_1, \dots, u_n) \in \mathcal{D}$ ,*

$$z^T D^2 h(U) A(U) z \geq \alpha \sum_{i=1}^n u_i^{2m_i-2} z_i^2.$$

Let  $U^0 \in L^1(\Omega; \mathcal{D})$  such that  $w^0 := Dh(U^0) \in L^\infty(\Omega; \mathbb{R}^n)$ . Then, there exists a weak solution  $U$  with initial condition  $U^0$  to

$$\begin{cases} \partial_t U - \nabla \cdot (A(U) \nabla U) = 0 & \text{on } (0, T) \times \Omega, \\ (A(U) \nabla U) \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (10)$$

such that for almost all  $(t, x) \in (0, T) \times \Omega$ ,  $U(t, x) \in \overline{\mathcal{D}}$  with

$$U \in L^2_{\text{loc}}((0, T); H^1(\Omega; \mathbb{R}^n)) \quad \text{and} \quad \partial_t U \in L^2_{\text{loc}}((0, T); (H^1(\Omega; \mathbb{R}^n))').$$

The proof of Theorem 2.1 relies on the fact that a system of the form (10) with a matrix-valued field  $A$  satisfying conditions (H1)-(H2) exhibits a (formal) gradient flow structure, which we detail below for the particular case of (7) with  $A$  defined by (9).

To this end, let us introduce the entropy density  $h$  given by

$$h : \begin{cases} \overline{\mathcal{D}} & \rightarrow & \mathbb{R} \\ U := (u_i)_{1 \leq i \leq n} & \mapsto & \sum_{i=1}^n u_i \log u_i + (1 - \sum_{i=1}^n u_i) \log(1 - \sum_{i=1}^n u_i). \end{cases} \quad (11)$$

and the corresponding entropy functional

$$\mathcal{E} : \begin{cases} L^\infty(\Omega, \overline{\mathcal{D}}) & \rightarrow & \mathbb{R} \\ U & \mapsto & \int_{\Omega} h(U(x)) dx. \end{cases} \quad (12)$$

It is proved in Lemma 2.3 of [14] that the function  $h$  defined by (11) satisfies conditions (H1) and (H2) of Theorem 2.1 for the matrix-valued function  $A$  defined by (9) with  $m_i = \frac{1}{2}$  for every  $1 \leq i \leq n$  and  $\alpha = \min_{1 \leq i \neq j \leq n} K_{ij}$ . Furthermore, we can rewrite the system (7) as

$$\begin{aligned} \partial_t U - \nabla \cdot (A(U)(D^2 h(U))^{-1} \nabla D\mathcal{E}(U)) &= 0 \quad \text{on } (0, T) \times \Omega, \\ (A(U)(D^2 h(U))^{-1} \nabla D\mathcal{E}(U)) \cdot \mathbf{n} &= 0 \quad \text{on } (0, T) \times \partial\Omega, \\ U(0, x) &= U^0(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

In this formulation, it becomes clear that the entropy functional  $\mathcal{E}$  is a Lyapunov function for system (7).

The existence of weak solutions to (7) satisfying  $U(t, x) \in \overline{\mathcal{D}}$  almost everywhere is then a direct consequence of Theorem 2 of [22] and Lemma 2.3 of [14]. More precisely, we have the following proposition

**Proposition 2.2** (Existence of weak solutions). *Let  $u^0 \in L^1(\Omega; \mathcal{P})$  and  $U^0 := (u_1^0, \dots, u_n^0)$ . Let us assume in addition that  $w^0 := Dh(U^0) \in L^\infty(\Omega; \mathbb{R}^n)$  with  $h$  defined by (11). Then, there exists a weak solution  $u$  with initial condition  $u^0$  to (4) such that*

- (i)  $u \in L^2_{\text{loc}}((0, T); H^1(\Omega; \mathbb{R}^{n+1}))$  and  $\partial_t u \in L^2_{\text{loc}}((0, T); (H^1(\Omega; \mathbb{R}^{n+1}))')$ ;
- (ii) for almost all  $(t, x) \in (0, T) \times \Omega$ ,  $u(t, x) \in \overline{\mathcal{P}}$ .

## 2.2 Main results

The aim of this work is to prove the existence of strong solutions to system (7) satisfying (2) under additional assumptions on the cross-diffusion coefficients  $(K_{ij})_{0 \leq i \neq j \leq n}$ . Such a result holds for arbitrary dimension  $d \in \mathbb{N}^*$ . For the particular case when  $d = 1$ , we can also prove a weak-strong stability result which implies that there exists a unique weak solution to system (4) satisfying (2) and that this solution is strong.

Before stating our main results, let us make a preliminary remark on the no-flux boundary conditions imposed on  $U$  in (7) which will be useful in the sequel. It is shown in [19, Lemma 5] that the matrix  $A(U)$  is invertible for all  $U \in \overline{\mathcal{D}}$ . Besides, for all  $U \in L^1(\Omega; \overline{\mathcal{D}})$ ,  $(A(U)\nabla U) \cdot \mathbf{n} = A(U)(\nabla U \cdot \mathbf{n})$  on  $\partial\Omega$ . This implies that a solution  $U$  to (7) is equivalently a solution to the system

$$\begin{cases} \partial_t U - \nabla \cdot (A(U)\nabla U) = 0 & \text{on } (0, T) \times \Omega, \\ \nabla U \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (13)$$

and, denoting by  $u_0 := 1 - \sum_{i=1}^n u_i$ ,  $u := (u_0, \dots, u_n)$  is then equivalently a solution to

$$\begin{cases} \partial_t u_i - \nabla \cdot \left[ \sum_{j=0, j \neq i}^n K_{ij} (u_j \nabla u_i - u_i \nabla u_j) \right] = 0 & \text{in } (0, T) \times \Omega, \\ \nabla u_i \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad i = 0, \dots, n. \quad (14)$$

Proving the existence of strong solutions to system (7) is then equivalent to proving the existence of strong solutions to system (14) and it will be more convenient for our analysis to consider the latter formulation in the sequel.

To obtain this strong existence result, we make an additional assumption on the cross-diffusion coefficients  $(K_{ij})_{0 \leq i \neq j \leq n}$  which we detail hereafter. For all  $0 \leq i \leq n$ , let

$$K_i^+ := \max_{0 \leq j \neq i \leq n} K_{ij}, \quad K_i^- := \min_{0 \leq j \neq i \leq n} K_{ij}, \quad K_i := \frac{K_i^+ + K_i^-}{2} \quad \text{and} \quad \kappa_i := \frac{K_i^+ - K_i^-}{2}. \quad (15)$$

Let us also denote by  $\kappa := \max_{0 \leq i \leq n} \kappa_i$ .

The additional assumption which we make from now on and in all the sequel reads as follows:

**Assumption 2.**  $\min_{0 \leq i \leq n} K_i > 2n\kappa$ .

In other words, Assumption 2 means that all the coefficients  $K_{ij}$  should be sufficiently close to one another. The motivation for considering such a situation stems from the following observation: if there exists a constant  $K > 0$  such that for all  $0 \leq i \neq j \leq n$ ,  $K_{ij} = K$ , then  $\kappa = 0$  and system (14) boils down to a system of  $n + 1$  independent heat equations for which the existence and uniqueness of strong solutions satisfying (2) is obvious.

We finally mention that our assumptions do not imply that the system (14) has a positive definite diffusion matrix for  $u \in \bar{\mathcal{P}}$ . Indeed, in system form we have

$$\partial_t u = D(u)\Delta u$$

where for all  $u := (u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ ,  $D(u) := (D(u)_{ij})_{0 \leq i, j \leq n} \in \mathbb{R}^{(n+1) \times (n+1)}$  is defined by

$$D(u)_{ii} = \sum_{j \neq i} K_{ij} u_j, \quad D(u)_{ij} = -K_{ij} u_i \quad \text{for all } 0 \leq i \neq j \leq n.$$

Now consider the simple case where  $u_0 = 1$  and  $u_i = 0$  for  $1 \leq i \leq n$ . Then,

$$D(u) := \begin{pmatrix} 0 & -K_{12} & -K_{13} & \cdots & -K_{1n} \\ 0 & K_{12} & 0 & \cdots & 0 \\ \vdots & 0 & K_{13} & 0 & \vdots \\ & & & \ddots & \\ 0 & \cdots & \cdots & 0 & K_{1n} \end{pmatrix}.$$

The first  $2 \times 2$  submatrix of the symmetrized part of  $D(u)$  is

$$\begin{pmatrix} 0 & -K_{12}/2 \\ -K_{12}/2 & K_{12} \end{pmatrix}$$

which has one positive and one negative eigenvalue. Hence, the symmetrized part of  $D(u)$  cannot be positive semidefinite. This prevents a straightforward analysis of the model and needs some refined argument using the structure of the equations on the set  $\bar{\mathcal{P}}$ , which is detailed below.

We are now in position to state our two main results.

**Theorem 2.3.** (*Existence of strong solutions in arbitrary dimension*) *Let us assume that Assumptions 1 and 2 hold. Then, for every initial datum  $u^0 \in [H^1(\Omega)]^{n+1}$ , with  $u^0(x) \in \bar{\mathcal{P}}$  for almost all  $x \in \Omega$ , there exists a strong solution  $u$  to (14) (or equivalently to (4)) such that*

- (i)  $u \in [L^2((0, T), H^2(\Omega)) \cap H^1((0, T), L^2(\Omega))]^{n+1}$ ;
- (ii)  $u(t, x) \in \bar{\mathcal{P}}$  for almost all  $(t, x) \in (0, T) \times \Omega$ .

**Theorem 2.4.** (*Weak-strong stability estimate in  $d = 1$* ) *Let us assume that Assumptions 1 and 2 hold. Let  $\tilde{u}$  be a weak solution to (14) (or equivalently to (4)) in the sense of Proposition 2.2, and let  $u$  be a strong solution in the sense of Theorem 2.3 which satisfies in addition the property that  $\nabla u \in L^2(0, T; L^\infty(\Omega))^n$ .*

*Then, there exists a constant  $C > 0$  such that the following stability estimate holds for all  $0 < t \leq T$ :*

$$\|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{L^2(\Omega)}^2 \leq e^{C\|\nabla u\|_{L^2(0, t; L^\infty(\Omega))}^2} \|u(0, \cdot) - \tilde{u}(0, \cdot)\|_{L^2(\Omega)}^2. \quad (16)$$

A direct corollary of Theorem 2.4 is the weak-strong uniqueness of solutions to (14) in dimension 1 for regular initial data, which can be stated as follows.

**Corollary 2.5.** *Let us assume that  $d = 1$  and let  $u^0 \in H^1(\Omega)^{n+1}$  such that  $u^0(x) \in \overline{\mathcal{P}}$  for almost all  $x \in \Omega$ . Let  $\tilde{u}$  be a weak solution to (14) (or equivalently to (4)) in the sense of Proposition 2.2 and  $u \in L^2((0, T), H^2(\Omega)) \cap H^1((0, T), L^2(\Omega))$  be a strong solution in the sense of Theorem 2.3. Then, if the corresponding initial data  $u(0, \cdot)$  and  $\tilde{u}(0, \cdot)$  agree a.e. on  $\Omega$ , we also have*

$$u = \tilde{u} \text{ a.e. in } \Omega \times (0, T).$$

The proof of this corollary stems from the fact that the injection  $H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$  is continuous in dimension 1, so that a strong solution  $u \in L^2((0, T), H^2(\Omega)) \cap H^1((0, T), L^2(\Omega))$  in the sense of Theorem 2.3, satisfies  $\nabla u \in L^2(0, T; L^\infty(\Omega))^n$ .

### 3 Proof of Theorem 2.3

The aim of this section is to prove Theorem 2.3 which states the existence of strong solutions to system (14). The proof is based on a fixed-point argument: We will show existence of strong solutions to a regularized, linearized system and subsequently apply Brouwer's fixed point theorem. For convenience let us define

$$W := L^2((0, T); H^2(\Omega)) \cap H^1((0, T); L^2(\Omega)).$$

#### 3.1 Auxiliary lemma

We start by stating an auxiliary result.

**Lemma 3.1.** *Let  $u \in W$ . Then, it holds that*

$$(i) \quad u \in \mathcal{C}([0, T]; H^1(\Omega));$$

(ii) *there exists a constant  $C > 0$  which only depends on  $T$  and  $\Omega$  such that*

$$\max_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^1(\Omega)} \leq C \left( \|u\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t u\|_{L^2(0, T; L^2(\Omega))} \right); \quad (17)$$

(iii) *if in addition  $\nabla u \cdot \mathbf{n} = 0$  a.e. in  $\partial\Omega \times (0, T)$ , the mapping  $(0, T) \ni t \mapsto \|\nabla u(t, \cdot)\|_{L^2(\Omega)}^2$  is absolutely continuous, with*

$$\frac{d}{dt} \|\nabla u(t, \cdot)\|_{L^2(\Omega)}^2 = -2 \langle \partial_t u(t, \cdot), \Delta u(t, \cdot) \rangle_{L^2(\Omega)}, \quad \text{for a.e. } t \in (0, T).$$

*Proof.* Items (i) and (ii) are direct applications of [Theorem 4, Section 5.9.2] of [15]. Let us now turn to the proof of (iii). We extend  $u$  by zero to a function  $\bar{u}$  defined for all  $t \in \mathbb{R}$  and define, for all  $\delta > 0$  and  $t \in \mathbb{R}$ ,  $u^\delta(t, \cdot) := \int_{\mathbb{R}} \eta_\delta(t - s) \bar{u}(s, \cdot) ds$ , where  $\eta_\delta$  is a standard mollifier. It then holds that  $u^\delta \in \mathcal{C}^\infty(0, T; H^2(\Omega))$  and that for all  $t \in (0, T)$ ,  $\nabla u^\delta(t, \cdot) \cdot \mathbf{n} = 0$  a.e. in  $\partial\Omega$ .

Let  $0 < t < T$ . Then, we have

$$\frac{d}{dt} \left( \|\nabla u^\delta(t, \cdot)\|_{L^2(\Omega)}^2 \right) = 2 \langle \partial_t \nabla u^\delta(t, \cdot), \nabla u^\delta(t, \cdot) \rangle_{L^2(\Omega)} = 2 \int_{\Omega} \partial_t \nabla u^\delta(t, \cdot) \cdot \nabla u^\delta(t, \cdot),$$

where  $\partial_t \nabla u^\delta$  is the weak time derivative of  $\nabla u^\delta$ . For all  $0 \leq t \leq T$ , the following convergence holds strongly in  $L^2(\Omega)$

$$\lim_{h \rightarrow 0} \frac{\nabla u^\delta(t+h, \cdot) - \nabla u^\delta(t, \cdot)}{h} = \partial_t \nabla u^\delta(t, \cdot),$$

and

$$\left\| \frac{\nabla u^\delta(t+h, \cdot) - \nabla u^\delta(t, \cdot)}{h} \right\|_{L^2(\Omega)} = \left\| \frac{1}{h} \int_t^{t+h} \partial_s \nabla u^\delta(s, \cdot) ds \right\|_{L^2(\Omega)} \leq \sup_{0 \leq s \leq T} \|\partial_s \nabla u^\delta(s, \cdot)\|_{L^2(\Omega)}. \quad (18)$$

Inequality (18) implies that the difference quotient  $\left( \frac{\nabla u^\delta(\cdot, t+h) - \nabla u^\delta(\cdot, t)}{h} \right)_{h>0}$  is uniformly bounded in  $L^2(\Omega)$  as  $h$  goes to 0, so that  $\left( \frac{\nabla u^\delta(\cdot, t+h) - \nabla u^\delta(\cdot, t)}{h} \cdot \nabla u^\delta \right)_{h>0}$  is uniformly bounded in  $L^1(\Omega)$ . As a consequence, applying Lebesgue's dominated convergence theorem, we obtain

$$2 \int_{\Omega} \partial_t \nabla u^\delta(t, \cdot) \cdot \nabla u^\delta(t, \cdot) = \lim_{h \rightarrow 0} \int_{\Omega} \frac{\nabla u^\delta(\cdot, t+h) - \nabla u^\delta(\cdot, t)}{h} \cdot \nabla u^\delta(\cdot, t).$$

Besides, since  $\nabla u^\delta(s, \cdot) \cdot \mathbf{n} = 0$  on  $\partial\Omega$  for all  $s \in (0, T)$ , it holds that for all  $h > 0$ ,

$$\int_{\Omega} \frac{\nabla u^\delta(\cdot, t+h) - \nabla u^\delta(\cdot, t)}{h} \cdot \nabla u^\delta = - \int_{\Omega} \frac{u^\delta(\cdot, t+h) - u^\delta(\cdot, t)}{h} \Delta u^\delta(\cdot, t).$$

Applying again Lebesgue's convergence theorem, we obtain

$$\lim_{h \rightarrow 0} \int_{\Omega} \frac{u^\delta(\cdot, t+h) - u^\delta(\cdot, t)}{h} \Delta u^\delta(\cdot, t) = \int_{\Omega} \partial_t u^\delta(\cdot, t) \Delta u^\delta(\cdot, t),$$

in  $L^1(\Omega)$ . Thus, for all  $\delta > 0$ , we have

$$\frac{d}{dt} \left( \|\nabla u^\delta(t, \cdot)\|_{L^2(\Omega)}^2 \right) = -2 \langle \partial_t u^\delta(t, \cdot), \Delta u^\delta(t, \cdot) \rangle_{L^2(\Omega)}. \quad (19)$$

As  $\delta$  goes to 0, the convergences  $u^\delta \rightarrow u$ ,  $\nabla u^\delta \rightarrow \nabla u$  and  $\Delta u^\delta \rightarrow \Delta u$  hold strongly in  $L^2(0, T; L^2(\Omega))$  (since  $u \in L^2((0, T); H^2(\Omega))$ ). We finally obtain the result by passing to the limit  $\delta \rightarrow 0$  in (19).  $\square$

## 3.2 Existence for a linear problem

To prove Theorem 2.3, we begin by proving the existence of a strong solution to a truncated linearized approximate problem, which we present hereafter.

Let us assume for now that there exists a smooth solution  $u := (u_0, \dots, u_n)$  to (7) satisfying  $\sum_{j=0}^n u_j = 1$ . Then, for all  $0 \leq i \leq n$ , the  $i^{\text{th}}$  component of (7) reads

$$\partial_t u_i = \sum_{j=0, j \neq i}^n K_{ij} (u_j \Delta u_i - u_i \Delta u_j), \quad (20)$$



Using the fact that  $\sum_{j=0}^n u_j = 1$ , equation (20) can be rewritten as

$$\partial_t u_i - K_i \Delta u_i = \sum_{j=0, j \neq i}^n (K_{ij} - K_i)(u_j \Delta u_i - u_i \Delta u_j), \quad (21)$$

where the positive constant  $K_i$  is defined in (15).

Let us denote by  $\sigma_0 := \frac{1}{2} \left( \frac{\min_{0 \leq i \leq n} K_i}{2n\kappa} - 1 \right)$ . From Assumption 2, it holds that  $\sigma_0 > 0$  and for all  $\sigma_0 \geq \sigma > 0$ ,

$$\min_{0 \leq i \leq n} K_i > 2n\kappa(1 + \sigma). \quad (22)$$

For all  $0 < \sigma \leq \sigma_0$ , we introduce the cut-off function  $f_\sigma$  defined as follows:

$$\forall x \in \mathbb{R}, \quad f_\sigma(x) = \begin{cases} e^{\frac{1}{\sigma^2} - \frac{1}{\sigma^2 - |x|^2}}, & -\sigma \leq x \leq 0, \\ 1, & 0 \leq x \leq 1, \\ e^{\frac{1}{\sigma^2} - \frac{1}{\sigma^2 - |x-1|^2}}, & 1 \leq x < 1 + \sigma, \\ 0, & \text{otherwise.} \end{cases} \quad (23)$$

The function  $f_\sigma$  satisfies the following properties:

(P1)  $f_\sigma \in \mathcal{C}^\infty(\mathbb{R})$ ,  $0 \leq f \leq 1$ ,  $f_\sigma = 1$  on  $[0, 1]$  and  $f_\sigma = 0$  on  $\mathbb{R} \setminus (-\sigma, 1 + \sigma)$ ;

(P2) for all  $x \in \mathbb{R}$ ,  $|xf_\sigma(x)| \leq 1 + \sigma$ ;

(P3) denoting by  $g_\sigma(x) := xf_\sigma(x)$  for all  $x \in \mathbb{R}$ , the function  $g_\sigma$  is Lipschitz continuous.

Let  $\tilde{u} := (\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_n) \in W^{n+1}$  and  $0 < \sigma \leq \sigma_0$ . We consider the following linear, regularized problem:

$$\begin{cases} \partial_t u_i - K_i \Delta u_i = \sum_{j=0, j \neq i}^n (K_{ij} - K_i)(f_\sigma(\tilde{u}_j) \tilde{u}_j \Delta u_i - f_\sigma(\tilde{u}_i) \tilde{u}_i \Delta u_j), & i = 0, \dots, n. \\ \nabla u_i \cdot \mathbf{n} = 0, \end{cases} \quad (24)$$

**Lemma 3.2** (Existence of a strong solution to the linearized problem). *For all  $0 < \sigma \leq \sigma_0$ ,  $\tilde{u} := (\tilde{u}_0, \dots, \tilde{u}_n) \in W^{n+1}$ , and  $u^0 := (u_0^0, \dots, u_n^0) \in [H^1(\Omega)]^{n+1}$ , there exists a unique solution  $u := (u_0, \dots, u_n) \in W^{n+1}$  to (24). In addition, the three following a priori estimates hold*

$$\sum_{i=0}^n \sup_{0 \leq t \leq T} \|\nabla u_i(\cdot, t)\|_{L^2(\Omega)}^2 + \sum_{i=0}^n \int_0^T \int_\Omega (\Delta u_i)^2 \leq C_0(\sigma), \quad (25)$$

$$\sup_{0 \leq s \leq T} \sum_{i=0}^n \|u_i(t, \cdot)\|_{L^2(\Omega)}^2 \leq C_1(\sigma), \quad (26)$$

$$\sum_{i=0}^n \|\partial_t u_i\|_{L^2(0, T; L^2(\Omega))}^2 \leq C_2(\sigma), \quad (27)$$

with

$$C_0(\sigma) := 2 \max \left( 1, \frac{1}{\min_{0 \leq i \leq n} K_i - 2n\kappa(1 + \sigma)} \right) \sum_{i=0}^n \|\nabla u_i^0\|_{L^2(\Omega)}^2, \quad (28)$$

$$C_1(\sigma) := e^{2n\kappa(1+\sigma)T} \left( \sum_{i=0}^n \|u_i^0\|_{L^2(\Omega)}^2 + 2n\kappa(1 + \sigma)C_0(\sigma) \right), \quad (29)$$

$$C_2(\sigma) := \left( \max_{0 \leq i \leq n} K_i + 2n\kappa(1 + \sigma) \right) C_0(\sigma). \quad (30)$$

*Proof.* Let us fix  $0 < \sigma \leq \sigma_0$  and  $u^0 := (u_0^0, \dots, u_n^0) \in (H^1(\Omega))^{n+1}$ .

*Step 1 (Regular data):* Let us first prove that the result holds for any regular function  $\tilde{u}$ . Let  $\tilde{u} := (\tilde{u}_0, \dots, \tilde{u}_n) \in \mathcal{C}^\infty([0, T] \times \bar{\Omega})$  and consider the system (24). Denoting by  $u := (u_0, \dots, u_n)$ , we can rewrite the system (24) as follows:

$$\partial_t u = (P - B(\tilde{u}))\Delta u \quad (31)$$

where  $P \in \mathbb{R}^{(n+1) \times (n+1)}$  is the diagonal matrix with entries  $(K_i)_{0 \leq i \leq n}$  and

$$B(\tilde{u}) := \begin{pmatrix} \sum_{j=0, j \neq 0}^n (K_{0j} - K_0) f_\sigma(\tilde{u}_j) \tilde{u}_j & \dots & -(K_{0n} - K_0) f_\sigma(\tilde{u}_0) \tilde{u}_0 \\ \vdots & \ddots & \vdots \\ -(K_{n0} - K_n) f_\sigma(\tilde{u}_n) \tilde{u}_n & \dots & \sum_{j=0, j \neq n}^n (K_{nj} - K_n) f_\sigma(\tilde{u}_j) \tilde{u}_j \end{pmatrix}.$$

For any  $\xi \in \mathbb{R}^{n+1}$ , we have, using property (P2) of the function  $f$ ,

$$\begin{aligned} \xi^T (P - B(\tilde{u})) \xi &= \sum_{i=0}^n K_i \xi_i^2 + \sum_{i=0}^n \sum_{j=0, i \neq j}^n (K_{ij} - K_i) (f_\sigma(\tilde{u}_j) \tilde{u}_j \xi_i^2 - f_\sigma(\tilde{u}_i) \tilde{u}_i \xi_i \xi_j) \\ &\geq \left( \min_{0 \leq i \leq n} K_i - n\kappa(1 + \sigma) \right) \|\xi\|_2^2 - \sum_{i=0}^n \sum_{j=0, i \neq j}^n (K_{ij} - K_i) f_\sigma(\tilde{u}_i) \tilde{u}_i \xi_i \xi_j. \end{aligned}$$

Besides, using again (P2), it holds that

$$\begin{aligned} \left| \sum_{i=0}^n \sum_{j=0, i \neq j}^n (K_{ij} - K_i) f_\sigma(\tilde{u}_i) \tilde{u}_i \xi_i \xi_j \right| &\leq \sum_{i=0}^n \sum_{j=0, i \neq j}^n \kappa_j |f_\sigma(\tilde{u}_i) \tilde{u}_i| |\xi_i \xi_j| \\ &\leq \frac{1}{2} \sum_{i=0}^n \sum_{j=0, i \neq j}^n \kappa_j |f_\sigma(\tilde{u}_i) \tilde{u}_i| (\xi_i^2 + \xi_j^2) \leq \frac{n}{2} \kappa(1 + \sigma) \|\xi\|_2^2 + \frac{n}{2} \kappa(1 + \sigma) \|\xi\|_2^2. \end{aligned}$$

We thus obtain

$$\xi^T (P - B(\tilde{u})) \xi \geq \left( \min_{0 \leq i \leq n} K_i - 2n\kappa(1 + \sigma) \right) \|\xi\|_2^2. \quad (32)$$

Due to (22), this estimate ensures that the linear system (31) (and thus (35)) is strongly parabolic (and in particular parabolic in the sense of Petrowskii), cf. [34, Ch. VII, Def. 1 and Def. 7]. Thus, [34, Theorem 10.1] ensures the existence of a unique strong solution  $U = (u_0, \dots, u_n) \in W^{n+1}$  to the equation.

On the one hand, multiplying (31) by the vector  $(-\Delta u_0, \dots, -\Delta u_n)^T$  and integrating over  $\Omega$ , we obtain that for almost all  $t \in (0, T)$ ,

$$-\sum_{i=0}^n (\partial_t u_i, \Delta u_i)_{L^2(\Omega)} + \left( \min_{0 \leq i \leq n} K_i - 2n\kappa(1 + \sigma) \right) \sum_{i=0}^n \int_{\Omega} (\Delta u_i)^2 dx \leq 0,$$

which implies, using Lemma 3.1 and integrating in time between 0 and  $t$ ,

$$\sum_{i=0}^n \|\nabla u_i(\cdot, t)\|_{L^2(\Omega)}^2 + \left( \min_{0 \leq i \leq n} K_i - 2n\kappa(1 + \sigma) \right) \sum_{i=0}^n \int_0^t \int_{\Omega} (\Delta u_i)^2 dx dt \leq \sum_{i=0}^n \|\nabla u_i^0\|_{L^2(\Omega)}^2.$$

We thus get

$$\sum_{i=0}^n \sup_{0 \leq t \leq T} \|\nabla u_i(\cdot, t)\|_{L^2(\Omega)}^2 + \left( \min_{0 \leq i \leq n} K_i - 2n\kappa(1 + \sigma) \right) \sum_{i=0}^n \int_0^T \int_{\Omega} (\Delta u_i)^2 dx dt \leq 2 \sum_{i=0}^n \|\nabla u_i^0\|_{L^2(\Omega)}^2, \quad (33)$$

which immediately yields (25). On the other hand, multiplying (31) by the vector  $(u_0, \dots, u_n)^T$  and integrating over  $\Omega$ , we obtain that for almost all  $t \in (0, T)$ ,

$$\begin{aligned} \sum_{i=0}^n \frac{d}{dt} \left( \|u_i(t, \cdot)\|_{L^2(\Omega)}^2 \right) + \sum_{i=0}^n \int_{\Omega} K_i |\nabla u_i|^2 dx &= \sum_{i=0}^n \sum_{j=0, j \neq i}^n (K_{ij} - K_i) \int_{\Omega} (f_{\sigma}(\tilde{u}_j) \tilde{u}_j u_i \Delta u_i - f_{\sigma}(\tilde{u}_i) \tilde{u}_i u_i \Delta u_j) dx \\ &\leq 2n\kappa(1 + \sigma) \sum_{i=0}^n \left( \|\Delta u_i(\cdot, t)\|_{L^2(\Omega)}^2 + \|u_i(t, \cdot)\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Applying Gronwall's lemma, we thus obtain that

$$\sup_{0 \leq s \leq T} \sum_{i=0}^n \|u_i(t, \cdot)\|_{L^2(\Omega)}^2 \leq e^{2n\kappa(1+\sigma)T} \left( \sum_{i=0}^n \|u_i^0\|_{L^2(\Omega)}^2 + 2n\kappa(1 + \sigma) \|\Delta u_i(\cdot, t)\|_{L^2(0, T; L^2(\Omega))}^2 \right), \quad (34)$$

which yields (26). Lastly, using (35), we obtain that

$$\sum_{i=0}^n \|\partial_t u_i\|_{L^2(0, T; L^2(\Omega))}^2 \leq \left( \max_{0 \leq i \leq n} K_i + 2n\kappa(1 + \sigma) \right) \sum_{i=0}^n \|\Delta u_i\|_{L^2(0, T; L^2(\Omega))}^2,$$

which immediately yields estimate (27).

*Step 2 (Irregular data):* Let us now assume that  $\tilde{u} := (\tilde{u}_0, \dots, \tilde{u}_n) \in W^{n+1}$ . Since  $\mathcal{C}^{\infty}([0, T] \times \overline{\Omega})^{n+1}$  is dense in  $W^{n+1}$ , there exists a family  $((\tilde{u}_0^{\delta}, \dots, \tilde{u}_n^{\delta}))_{\delta > 0} \subset \mathcal{C}^{\infty}([0, T] \times \overline{\Omega})^{n+1}$  such that for all  $0 \leq i \leq n$ ,  $(\tilde{u}_i^{\delta})_{\delta > 0}$  strongly converges to  $\tilde{u}_i$  in  $W$  as  $\delta$  goes to 0. For all  $\delta > 0$ , let us denote by  $(u_0^{\delta}, \dots, u_n^{\delta}) \in W^{n+1}$  the unique solution to

$$\begin{aligned} \partial_t u_i^{\delta}(x, t) - K_i \Delta u_i^{\delta} &= \sum_{j=0, j \neq i}^n (K_{ij} - K_i) (f_{\sigma}(\tilde{u}_j^{\delta}) \tilde{u}_j^{\delta} \Delta u_i^{\delta} - f_{\sigma}(\tilde{u}_i^{\delta}) \tilde{u}_i^{\delta} \Delta u_j^{\delta}), \quad i = 0, \dots, n. \\ \nabla u_i^{\delta} \cdot \mathbf{n} &= 0, \end{aligned} \quad (35)$$

with initial condition  $(u_0^0, \dots, u_n^0)$ . From Step 1, we have that for all  $\delta > 0$ ,

$$\begin{aligned} \sum_{i=0}^n \sup_{0 \leq t \leq T} \|\nabla u_i^\delta(\cdot, t)\|_{L^2(\Omega)}^2 + \sum_{i=0}^n \int_0^T \int_{\Omega} (\Delta u_i^\delta)^2 dxdt &\leq C_0(\sigma), \\ \sup_{0 \leq s \leq T} \sum_{i=0}^n \|u_i^\delta(t, \cdot)\|_{L^2(\Omega)}^2 &\leq C_1(\sigma), \\ \sum_{i=0}^n \|\partial_t u_i^\delta\|_{L^2(0, T; L^2(\Omega))}^2 &\leq C_2(\sigma), \end{aligned}$$

where  $C_0(\sigma)$ ,  $C_1(\sigma)$  and  $C_2(\sigma)$  are defined in (28), (29) and (30) respectively. We thus obtain that the sequence  $(u_0^\delta, \dots, u_n^\delta)_{\delta > 0}$  is thus bounded in  $W^{n+1}$ . Up to the extraction of a subsequence, there exists  $(u_0, \dots, u_n) \in W^{n+1}$  such that  $(u_i^\delta)_{\delta > 0}$  weakly converges in  $W$  to  $u_i$  for all  $0 \leq i \leq n$ . In addition, using property (P3), it holds that for all  $0 \leq i \leq n$ , the sequence  $(\tilde{u}_i^\delta f_\sigma(\tilde{u}_i^\delta))_{\delta > 0}$  strongly converges in  $L^2(0, T; L^2(\Omega))$  to  $\tilde{u}_i f_\sigma(\tilde{u}_i)$ .

We can thus pass to the limit  $\delta \rightarrow 0$  in (35) in the distributional sense, and we obtain that  $(u_0, \dots, u_n)$  is a solution to

$$\begin{cases} \partial_t u_i(x, t) - K_i \Delta u_i = \sum_{j=0, j \neq i}^n (K_{ij} - K_i)(f_\sigma(\tilde{u}_j) \tilde{u}_j \Delta u_i - f_\sigma(\tilde{u}_i) \tilde{u}_i \Delta u_j), & i = 0, \dots, n. \\ \nabla u_i \cdot \mathbf{n} = 0 \end{cases} \quad (36)$$

In particular, we obtain that  $\partial_t u_i^\delta$  weakly converges in  $L^2(0, T; L^2(\Omega))$  to  $\partial_t u_i$  as  $\delta$  goes to 0. Passing to the limit  $\delta \rightarrow 0$  for all  $0 \leq i \leq n$ , in the identity

$$u_i^\delta(\cdot, t) - u_i^0 = \int_0^t \partial_t u_i^\delta(\cdot, s) ds,$$

we obtain that  $u_i(0, \cdot) = u_i^0$ . Besides, one easily gets estimates (25), (26) and (27). The uniqueness of the solution in  $W^{n+1}$  to (24) is an immediate consequence of estimates (25), (26) and (27) and of the linearity of the system.  $\square$

*Remark 1 (Uniqueness).* While we expect uniqueness of strong solutions in any spatial dimension, we only obtain it in 1D as a consequence of our weak-strong stability result below. Indeed, trying to prove strong-strong uniqueness based on Gronwall's inequality does not seem to work in any  $L^p$  and while the estimates get better for larger  $p$ , even in the limiting case  $p = \infty$  (using Duhamel's formula), we do not have enough integrability to make the argument work.

### 3.3 Proof of Theorem 2.3

*Proof of Theorem 2.3.* Let  $u^0 \in H^1(\Omega)^{n+1}$  such that  $u^0(x) \in \bar{P}$  for almost all  $x \in \Omega$ .

Let  $\sigma \in (0, \sigma_0]$  and let us denote by  $\mathcal{M}_\sigma$  the set of functions  $(u_0, \dots, u_n) \in W^{n+1}$  satisfying (25), (26) and (27) with constants  $C_0(\sigma)$ ,  $C_1(\sigma)$  and  $C_2(\sigma)$  defined by (28), (29) and (30) respectively. For all  $(\tilde{u}_0, \dots, \tilde{u}_n) \in \mathcal{M}_\sigma$ , let us denote by  $S_\sigma((\tilde{u}_0, \dots, \tilde{u}_n)) := (u_0, \dots, u_n) \in W^{n+1}$ , where  $(u_0, \dots, u_n)$  is the unique strong solution of (24).

In view of Lemma 2.3, the operator  $S_\sigma : \mathcal{M}_\sigma \rightarrow W^{n+1}$  is well-defined and self-mapping, i.e.  $S_\sigma(\mathcal{M}_\sigma) \subset \mathcal{M}_\sigma$ . Moreover, due to the Aubin-Lions lemma [26, Theorem 5.1, p. 58], the set  $\mathcal{M}_\sigma$  is

a convex compact subset of  $L^2((0, T); L^2(\Omega))$ . Following the same lines as in Step 2 of the proof of Lemma 3.2, we obtain that  $S_\sigma$  is a continuous mapping with respect to the norm of  $L^2((0, T); L^2(\Omega))$ . More precisely, for all sequences  $(\tilde{u}_0^\delta, \dots, \tilde{u}_n^\delta)_{\delta>0} \subset \mathcal{M}_\sigma$  which strongly converge in  $L^2((0, T); L^2(\Omega))$  to some  $(\tilde{u}_0, \dots, \tilde{u}_n) \in \mathcal{M}_\sigma$ , we have that  $S_\sigma((\tilde{u}_0^\delta, \dots, \tilde{u}_n^\delta))$  strongly converges in  $L^2((0, T); L^2(\Omega))$ , as  $\delta$  goes to 0, to  $S_\sigma((\tilde{u}_0, \dots, \tilde{u}_n))$ .

Thus, we can apply Brouwer's fixed point theorem and conclude to the existence of a strong solution  $(u_0^\sigma, \dots, u_n^\sigma) \in W^{n+1}$  to the regularized system

$$\begin{cases} \partial_t u_i^\sigma - K_i \Delta u_i^\sigma = \sum_{j=0, j \neq i}^n (K_{ij} - K_i) (f_\sigma(u_j^\sigma) u_j^\sigma \Delta u_i^\sigma - f(u_i^\sigma) u_i^\sigma \Delta u_j^\sigma), \\ \nabla u_i^\sigma \cdot \mathbf{n} = 0, \\ u_i^\sigma(0, \cdot) = u_i^0, \end{cases} \quad i = 0, \dots, n, \quad (37)$$

which satisfies the a priori estimates

$$\begin{aligned} \sum_{i=0}^n \sup_{0 \leq t \leq T} \|\nabla u_i^\sigma(\cdot, t)\|_{L^2(\Omega)}^2 + \sum_{i=0}^n \int_0^T \int_\Omega (\Delta u_i^\sigma)^2 &\leq C_0(\sigma), \\ \sup_{0 \leq s \leq T} \sum_{i=0}^n \|u_i^\sigma(t, \cdot)\|_{L^2(\Omega)}^2 &\leq C_1(\sigma), \\ \sum_{i=0}^n \|\partial_t u_i^\sigma\|_{L^2(0, T; L^2(\Omega))}^2 &\leq C_2(\sigma), \end{aligned}$$

where  $C_0(\sigma)$ ,  $C_1(\sigma)$  and  $C_2(\sigma)$  are defined respectively in (28), (29) and (30). Since  $(C_0(\sigma))_{\sigma>0}$ ,  $(C_1(\sigma))_{\sigma>0}$  and  $(C_2(\sigma))_{\sigma>0}$  are bounded sequences as  $\sigma$  goes to 0, the sequence  $((u_0^\sigma, u_1^\sigma, \dots, u_n^\sigma))_{\sigma>0}$  is bounded in  $W^{n+1}$ . Thus, up to the extraction of a subsequence, there exists  $u := (u_0, \dots, u_n) \in W^{n+1}$  such that for all  $0 \leq i \leq n$ ,  $(u_i^\sigma)_{\sigma>0}$  weakly converges in  $W$  (and hence strongly converges in  $L^2(0, T; L^2(\Omega))$  and a.e.) to  $u_i$ .

For all  $x \in \mathbb{R}$ , let us denote by  $x^\diamond := \max(0, \min(1, x))$ . The sequence of functions  $(|u_i^\sigma f_\sigma(u_i^\sigma) - u_i^\diamond|^2)_{\sigma>0}$  converges a.e. to 0 as  $\sigma$  goes to 0. Besides, for all  $0 < \sigma \leq \sigma_0$ ,

$$|u_i^\sigma f_\sigma(u_i^\sigma) - u_i^\diamond|^2 \leq 2|u_i^\sigma f_\sigma(u_i^\sigma)|^2 + 2|u_i^\diamond|^2 \leq 2(1 + \sigma)^2 + 2|u_i^\diamond|^2 \leq 2(1 + \sigma_0)^2 + 2|u_i^\diamond|^2.$$

Applying Lebesgue's dominated convergence theorem, we obtain that the sequence  $(u_i^\sigma f_\sigma(u_i^\sigma))_{\sigma>0}$  strongly converges in  $L^2(0, T; L^2(\Omega))$  to  $u_i^\diamond$ . We can thus pass to the limit  $\sigma \rightarrow 0$  in (37) and obtain that  $(u_0, \dots, u_n)$  is a solution in  $W^{n+1}$  to

$$\begin{cases} \partial_t u_i - K_i \Delta u_i = \sum_{j=0, j \neq i}^n (K_{ij} - K_i) (u_j^\diamond \Delta u_i - u_i^\diamond \Delta u_j), \\ \nabla u_i \cdot \mathbf{n} = 0, \\ u_i(0, \cdot) = u_i^0 \end{cases} \quad i = 0, \dots, n. \quad (38)$$

To end the proof, it remains to show that  $u_i \geq 0$  almost everywhere in  $(0, T) \times \Omega$  for all  $0 \leq i \leq n$ . For all  $x \in \mathbb{R}$ , let us denote by  $x^- = \min(0, x)$  the negative part of  $x$ . Since for all  $0 \leq i \leq n$ ,  $u_i \in W$ , its negative part  $u_i^-$  belongs to  $L^2(0, T; H^1(\Omega))$ . Multiplying the  $i^{\text{th}}$  equation of (38) by  $u_i^-$  and integrating over  $\Omega$ , we obtain

$$\int_\Omega \partial_t u_i u_i^- + K_i \int_\Omega \nabla u_i \cdot \nabla u_i^- = \sum_{j=0, j \neq i}^n (K_{ij} - K_i) (u_j^\diamond \nabla u_i \cdot \nabla u_i^- - u_i^\diamond u_i^- \Delta u_j).$$

It holds that  $\partial_t u_i u_i^- = \partial_t u_i^- u_i^-$ ,  $\nabla u_i \cdot \nabla u_i^- = |\nabla u_i^-|^2$  and  $u_i^\circ u_i^- = 0$ , so that

$$\frac{d}{dt} \left( \|u_i^-\|_{L^2(\Omega)}^2 \right) + (K_i - n\kappa) \int_{\Omega} |\nabla u_i^-|^2 \leq 0.$$

Since  $K_i > n\kappa$ , this implies that for all  $t \in (0, T)$ ,  $\|u_i^-(t, \cdot)\|_{L^2(\Omega)}^2 \leq \|u_i^-(0, \cdot)\|_{L^2(\Omega)}^2$  and as  $u_i^0$  is a non-negative function, this implies that  $u_i^-(t, x) = 0$  for almost all  $(t, x) \in (0, T) \times \Omega$ . For all  $0 \leq i \leq n$  and almost all  $(t, x) \in (0, T) \times \Omega$ ,  $u_i(t, x) \geq 0$  and  $\sum_{i=0}^n \partial_t u_i(t, x) = 0$ , hence  $\sum_{i=0}^n u_i(t, x) = \sum_{i=0}^n u_i^0(x) = 1$ . Thus,  $u(t, x) \in \overline{\mathcal{P}}$  almost everywhere.  $\square$

## 4 Weak strong stability

This section is devoted to the proof of Theorem 2.4, which provides a weak-strong stability result provided that there exists a strong solution  $u$  to the system of interest which satisfies the additional regularity property  $\nabla u \in L^2(0, T; L^\infty(\Omega))$ .

*Proof of Theorem 2.4.* Let  $K := \min_{0 \leq i \leq n} K_i$ . We start by rewriting the  $i^{\text{th}}$  component of (14) as

$$\int_{\Omega} \partial_t u_i \varphi \, dx + K \int_{\Omega} \nabla u_i \cdot \nabla \varphi \, dx = \int_{\Omega} \left[ \sum_{j=1, i \neq j}^n (K_{ij} - K)(u_j \nabla u_i - u_i \nabla u_j) \right] \cdot \nabla \varphi \, dx,$$

for all  $\varphi \in H^1(\Omega)$ . Denoting by

$$D(v) := \begin{pmatrix} \sum_{j=1}^n (K_{0j} - K)v_j & \dots & -(K_{0n} - K)v_0 \\ \vdots & \ddots & \vdots \\ -(K_{n0} - K)v_n & \dots & \sum_{j=0}^{n-1} (K_{nj} - K)v_j \end{pmatrix}.$$

for all  $v := (v_i)_{0 \leq i \leq n} \in \overline{\mathcal{P}}$ , we obtain that

$$\int_{\Omega} \partial_t u \Phi \, dx + \int_{\Omega} K \nabla u \cdot \nabla \Phi \, dx = \int_{\Omega} D(u) \nabla u \cdot \nabla \Phi \, dx, \text{ for all } \Phi \in [H^1(\Omega)]^{n+1}, \quad (39)$$

Since we know that  $\sum_{i=0}^n u_i = 1$  and that  $u_i \geq 0$  for  $i = 0, \dots, n$ , we immediately obtain that

$$\|D(u)\|_{L^\infty(\Omega)} \leq 2n\kappa,$$

in the sense of the spectral matrix norm. In addition,  $D : \overline{\mathcal{P}} \rightarrow \mathbb{R}^{(n+1) \times (n+1)}$  is Lipschitz continuous, with Lipschitz constant  $2n\kappa$ .

Now we consider the difference of the respective weak formulations (39) for  $u$  and  $\tilde{u}$  and obtain

$$\int_{\Omega} \partial_t (u - \tilde{u}) \Phi \, dx - K \int_{\Omega} (\nabla u - \nabla \tilde{u}) \cdot \nabla \Phi \, dx = \int_{\Omega} [D(u) \nabla u - D(\tilde{u}) \nabla \tilde{u}] \cdot \nabla \Phi \, dx.$$

Taking  $\Phi = (u - \tilde{u})(t, \cdot)$  (which belongs to  $H^1(\Omega)$  for almost all  $t \in (0, T)$ ) yields

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \|u - \tilde{u}\|_{L^2(\Omega)}^2 + K \|\nabla(u - \tilde{u})\|_{L^2(\Omega)}^2 \\ &= - \int_{\Omega} (D(u) - D(\tilde{u})) \nabla u \cdot \nabla(u - \tilde{u}) \, dx - \int_{\Omega} D(\tilde{u}) \nabla(u - \tilde{u}) \cdot \nabla(u - \tilde{u}) \, dx \end{aligned}$$

Using the fact that  $\|D(\tilde{u})\|_{L^\infty(\Omega)} \leq 2n\kappa$  on the second term of the right hand side, we obtain

$$\frac{d}{dt} \frac{1}{2} \|u - \tilde{u}\|_{L^2(\Omega)}^2 + (K - 2n\kappa) \|\nabla(u - \tilde{u})\|_{L^2(\Omega)}^2 \leq - \int_{\Omega} (D(u) - D(\tilde{u})) \nabla u \cdot \nabla(u - \tilde{u}) \, dx.$$

Since  $d = 1$ , it holds that  $L^\infty(\Omega) \subset H^1(\Omega)$  with continuous injection, which implies that  $\|\nabla u\|_{L^\infty(\Omega)} \in L^2(0, T)$  (since  $u \in L^2((0, T), H^2(\Omega))$ ). Thus, applying the weighted Young's inequality with  $0 < \epsilon < (K - 2n\kappa)$  and using the Lipschitz continuity of  $D$  yield

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|u - \tilde{u}\|_{L^2(\Omega)}^2 + (K - 2n\kappa - \epsilon) \|\nabla(u - \tilde{u})\|_{L^2(\Omega)}^2 &\leq \frac{1}{4\epsilon} \|(D(u) - D(\tilde{u})) \nabla u\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{4\epsilon} \|\nabla u\|_{L^\infty(\Omega)}^2 \|D(u) - D(\tilde{u})\|_{L^2(\Omega)}^2 \\ &\leq \frac{2n\kappa}{4\epsilon} \|\nabla u\|_{L^\infty(\Omega)}^2 \|u - \tilde{u}\|_{L^2(\Omega)}^2. \end{aligned}$$

Applying the differential form of the Gronwall lemma then implies that there exists  $C' > 0$  such that for all  $t \in (0, T)$ ,

$$\|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{L^2(\Omega)}^2 \leq e^{C' \|\nabla u\|_{L^2((0, t), L^\infty(\Omega))}^2} \|u(0, \cdot) - \tilde{u}(0, \cdot)\|_{L^2(\Omega)}^2,$$

with  $C' = \frac{2n\kappa}{4\epsilon}$ . Hence the result.  $\square$

*Remark 2.* Let us remark that in dimension one, a strong solution  $u$  in the sense of Theorem 2.3 necessarily satisfies  $\nabla u \in L^2(0, T; L^\infty(\Omega))$  since the injection  $H^2(\Omega) \hookrightarrow W^{1, \infty}(\Omega)$  is continuous. To extend this results in higher dimension, one would need to prove the existence of solutions with this additional regularity property, for instance with more regular initial data.

## Appendix: Microscopic interpretation

Following [4], we briefly describe a lattice based modelling approach and a formal way to obtain a (7) in the limit. We start with a one-dimensional lattice on which particles of  $i = 1, \dots, n$  species can jump to neighbouring sites. Let  $\mathcal{T}_h$  denote an equidistant grid of mesh size  $h$ , where a cell is either empty or can be occupied by at most one particle. We denote the probability to find a particle of species  $i$  at location  $x$  and time  $t$  by

$$c_i(x, t) = P(\text{particle of species } i \text{ at position } x \text{ at time } t),$$

and assume that the motion of these particles is due to two different effects: Diffusion and exchange (switching) of particles of different species. To this end, we introduce the rates

$$\Pi_{c_i}^+ = P(\text{jump of } c_i \text{ from position } x \text{ to } x + h \text{ in } (t, t + \Delta t)) \quad (40)$$

$$= K_{i0}(1 - \rho) + \sum_{j=1, i \neq j}^n K_{ij} c_j, \quad (41)$$

$$\Pi_{c_i}^- = P(\text{jump of } c_i \text{ from position } x \text{ to } x - h \text{ in } (t, t + \Delta t))$$

$$= K_{i0}(1 - \rho) + \sum_{j=1, i \neq j}^n K_{ij} c_j. \quad (42)$$

Here  $K_{i0}$  is a diffusion coefficient which controls the tendency of a particle to jump to a neighboring site. Since we restrict to at most one particle per site, this has to be modified by a factor of  $(1 - \rho)$ , i.e. the particle can only jump if the target site is empty. On the other hand, in order to exchange places with a particle from a different species, the target site has to be occupied and thus, for the second term we have to multiply the rate  $K_{ij}$  with  $c_j$ .

Now we consider the following cases: If  $K_{i0} \gg K_{ij}$ , then the probability of switching is small compared to that of diffusion and the effect of size exclusion will be essential. If, on the other hand  $K_{i0} \ll K_{ij}$ , switching will dominate and size exclusion will not play a role anymore. Note that in this case,  $\rho$ , which is the sum of all densities, remains constant.

Our subsequent analysis deals with the case when  $K_{i0} \approx K_{ij}$ , which is the most interesting. In fact, let us rewrite (40) as follows:

$$\begin{aligned} \Pi_{c_i}^+ &= K_{i0}(1 - \sum_{j=1, i \neq j}^n c_j - c_i) + \sum_{j=1, i \neq j}^n K_{ij}c_j, \\ &= K_{i0}(1 - c_i) + \sum_{j=1, i \neq j}^n (K_{ij} - K_{i0})c_j. \end{aligned}$$

Now if  $K_{i0} \approx K_{ij}$ , the switching will effectively anneal the size exclusion effect. In other words, it does not make a difference whether a target site is occupied by a particle of species  $j$  or if it is empty since in both cases, the particle at the source site can reach this target: Either by jumping to the empty cell or by switching positions. The resulting PDE can be written as

$$\begin{aligned} \partial_t c_i &= \nabla \cdot (K_{i0}((1 - c_i)\nabla c_i + c_i\nabla c_i) + \sum_{j=1, i \neq j}^n (K_{i0} - K_{ij})(c_j\nabla c_i - c_i\nabla c_j)) \\ &= \nabla \cdot (K_{i0}c_i + \sum_{j=1, i \neq j}^n (K_{i0} - K_{ij})(c_j\nabla c_i - c_i\nabla c_j)), \quad i = 1, \dots, n. \end{aligned}$$

which reveals that we are dealing with a perturbation of the heat equations, as already entailed in (3) in the introduction.

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